

# Changepoint Problem in Quantum Setting

Daiki Akimoto<sup>1</sup> and Masahito Hayashi<sup>1,2,\*</sup>

<sup>1</sup>Graduate School of Information Sciences, Tohoku University, Aoba-ku, Sendai, 980-8579, Japan

<sup>2</sup>Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, 117542, Singapore

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In the changepoint problem, we determine when the distribution observed has changed to another one. We expand this problem to the quantum case where copies of an unknown pure state are being distributed. We study the fundamental case, which has only two candidates to choose. This problem is equal to identifying a given state with one of the two unknown states when multiple copies of the states are provided. In this paper, we assume that two candidate states are distributed independently and uniformly in the space of the whole pure states. The minimum of the averaged error probability is given and the optimal POVM is defined as to obtain it. Using this POVM, we also compute the error probability which depends on the inner product. These analytical results allow us to calculate the value in the asymptotic case, where this problem approaches to the usual discrimination problem.

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## I. INTRODUCTION

The changepoint problem, which is studied in many fields, originally arose out of considerations of quality control. When a process is “in control,” products are produced according to some rule. At the unknown point the process jumps “out of control” and ensuing products are produced according to another rule. It is necessary to determine the changepoint.

In the classical case, we observe sequentially a discrete series of independent observations  $X_0, X_1, \dots$  whose distribution possibly changes at an unknown point in time. It is assumed that independent random variables  $X_0, X_1, \dots, X_{\nu-1}$  are each distributed according to a distribution and the remaining independent random variables  $X_{\nu}, X_{\nu+1}, \dots$  are each distributed according to another distribution. Our purpose is to detect the changepoint “ $\nu$ ” [1].

We extend this problem to the quantum setting where copies of an unknown pure state are being distributed in discrete time. Consider now the device distributing copies of an unknown pure state. This device has the unknown changepoint and distributes copies of another unknown pure state after the changepoint. Our goal is estimating the changepoint by observing all copies the device distributed.

In this paper, we deal with the fundamental case, that is, we have only two candidates for the changepoint. We assume that the device distributes unknown pure states  $\rho_t$  on the  $d$ -dimensional space for the discrete time  $t = 0, 1, 2, \dots, t_3$ . The state  $\rho_t$  is changed at the changepoint  $t_c$ . That is, the states  $\rho_0, \dots, \rho_{t_c-1}$  are identical, and the other states  $\rho_{t_c}, \dots, \rho_{t_3}$  are also identical. Further, we assume that there are two candidates of the changepoint  $t_1$  and  $t_2$ .

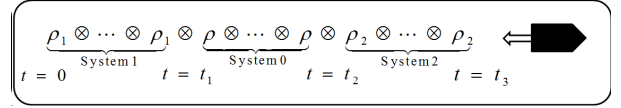


FIG. 1: Systems 1, 0, and 2

In order to analyze this problem, we introduce System 1, System 0 and System 2 to denote the systems that are the composite systems corresponding to the time period  $0 \leq t < t_1$ ,  $t_1 \leq t < t_2$  and  $t_2 \leq t \leq t_3$ , respectively, as is explained in Fig. 1. Our task is then choosing the correct changepoint  $t_c$  by using all three systems. This problem is equal to decide whether the state in System 0 coincides with the state in System 1 or the state in System 2.

We derive the optimal POVM and the minimum of the averaged error probability. This minimum probability is already obtained when the numbers of copies of states in System 1 and System 2 are the same. Sentis et al. studied this problem in the qubits case, i.e., in the case of  $d = 2$  [2] and A. Hayashi et al. discussed it when System 0 only has one copy [3]. Our result concerns the general case that has no restriction for the numbers of copies and the dimension in systems.

We also compute the error probability under the application of our optimal POVM, not averaged, which depends on the inner product of two states in System 1 and System 2. As the numbers of copies in three systems approach infinity, this error probability clearly approaches 0 exponentially unless the inner product is 1. Hence, the convergence speed can be measured by the exponential decreasing rate. The exponential decreasing rate seems related to the quantum Chernoff bound, which gives the optimal exponential decreasing rate of two state discrimination when a large number of copies of the unknown state are available [5], while this relation has not been discussed. In this paper, using the above analytical result, we calculate the exponential decreasing rate and clarify

\*Electronic address: hayashi@math.is.tohoku.ac.jp

the relationship with the quantum Chernoff bound.

The paper is organized as follows. In the next Section we give the optimal strategy for the minimum averaged error probability. The optimal POVM is described by using the representation theory for easy calculation in the following sections. In Section III we obtain the minimum error probability represented as a function of the number of copies in each three systems and the dimension of the state space. Using the optimal POVM, in Section IV we compute the error probability which depends on the inner product of two states in System 1 and System 2. In Section V we consider the asymptotic behaviors of the minimum averaged error probability in several scenarios. In Section VI, we finally compute the convergence speed of the error probability. Some brief conclusions follow and we end up with a technical appendix.

## II. THE OPTIMAL POVM

In order to treat the decision problem of the change-point, we denote the numbers  $t_1, t_2 - t_1, t_3 - t_2 + 1$  by  $N_1, M, N_2$ , respectively. Then, we have  $N_1$  ( $N_2$ ) copies of unknown pure states in System 1 (2). In the following, we denote the unknown state on System 1 (2) by  $\rho_1$  ( $\rho_2$ ), which is a pure state on the  $d$  dimensional vector space  $\mathbb{C}^d$ . System 0 has  $M$  copies of the unknown state  $\rho$  that is guaranteed to be either one of  $\rho_1$  and  $\rho_2$ . Note that we assume that  $N_1 \leq N_2$  which loses no generality of this problem. Our purpose is to identify the state  $\rho$  with one of the two states by using all systems. This is equal to distinguishing two states,  $\rho_1^{\otimes N_1} \otimes \rho_1^{\otimes M} \otimes \rho_2^{\otimes N_2}$  and  $\rho_1^{\otimes N_1} \otimes \rho_2^{\otimes M} \otimes \rho_2^{\otimes N_2}$ , which are assumed to occur with equal probability.

In this decision problem, we apply two-valued POVM  $\{E_1, E_2\}$ , in which  $E_1$  ( $E_2$ ) corresponds to the decision  $t_c = t_1$  ( $t_c = t_2$ ). Since  $E_1 = I - E_2$ , our POVM can be described by a Hermitian matrix  $0 \leq E_2 \leq I$ , where  $I$  is the unit matrix. Then, the error probability is defined as

$$\begin{aligned} p_{M,N_1,N_2}(\rho_1, \rho_2, E_2) \\ &\equiv \frac{1}{2} \text{Tr}[(\rho_1^{\otimes N_1} \otimes \rho_2^{\otimes M+N_2})E_2] + \frac{1}{2} \text{Tr}[(\rho_1^{\otimes M+N_1} \otimes \rho_2^{\otimes N_2})E_1] \\ &= \frac{1}{2} \text{Tr}[(\rho_1^{\otimes N_1} \otimes \rho_2^{\otimes M+N_2})E_2] \\ &\quad + \frac{1}{2} \text{Tr}[(\rho_1^{\otimes M+N_1} \otimes \rho_2^{\otimes N_2})(I - E_2)]. \end{aligned} \quad (1)$$

Now, we assume that  $\rho_1$  and  $\rho_2$  are independently distributed according to the unitary invariant distribution  $\mu_{\Theta_d}$  on the set  $\Theta_d$  of pure states on the  $d$  dimensional vector space  $\mathbb{C}^d$ . By using the POVM  $\{I - E_2, E_2\}$ , we can define the averaged error probability as

$$\begin{aligned} \bar{p}_{M,N_1,N_2}(E_2) \\ &\equiv \int_{\Theta_d} \int_{\Theta_d} p_{M,N_1,N_2}(\rho_1, \rho_2, E_2) \mu_{\Theta_d}(d\rho_1) \mu_{\Theta_d}(d\rho_2). \end{aligned} \quad (2)$$

Here it is very helpful to use the following formula for the integral of the tensor product of  $L$  identically prepared pure states [3]:

$$\int_{\Theta_d} \sigma^{\otimes L} \mu_{\Theta_d}(d\sigma) = \frac{I_L}{\text{Tr}[I_L]}, \quad (3)$$

where  $\sigma$  is a pure state and  $I_L$  is the projector onto the totally symmetric subspace of  $(\mathbb{C}^d)^{\otimes L}$ .

By using this formula, the averaged error probability reads

$$\begin{aligned} \bar{p}_{M,N_1,N_2}(E_2) \\ &= \frac{1}{2} \left[ 1 + \text{Tr} \left[ \left( \frac{I_{N_1} \otimes I_{M+N_2}}{A_1} - \frac{I_{M+N_1} \otimes I_{N_2}}{A_2} \right) E_2 \right] \right], \end{aligned} \quad (4)$$

where  $A_1$  and  $A_2$  are defined as follows:

$$\begin{aligned} A_1 &\equiv \text{Tr}[I_{N_1}] \text{Tr}[I_{M+N_2}] \\ &= \binom{N_1 + d - 1}{d - 1} \binom{M + N_2 + d - 1}{d - 1}. \end{aligned} \quad (5)$$

$$\begin{aligned} A_2 &\equiv \text{Tr}[I_{M+N_1}] \text{Tr}[I_{N_2}] \\ &= \binom{M + N_1 + d - 1}{d - 1} \binom{N_2 + d - 1}{d - 1}. \end{aligned} \quad (6)$$

Note that  $A_1 \leq A_2$ , and the equation holds if and only if  $N_1 = N_2$ .

Eq. (4) guarantees that the optimal strategy to minimize the averaged error probability is given by the Hermitian matrix

$$E_{M,N_1,N_2} \equiv \left\{ \frac{I_{N_1} \otimes I_{M+N_2}}{A_1} - \frac{I_{M+N_1} \otimes I_{N_2}}{A_2} < 0 \right\}, \quad (7)$$

where  $\{A < 0\}$  represents a projector onto the eigenspaces with negative eigenvalues of  $A$ . That is, plugging Eq. (7) into Eq. (4), one obtains the minimum averaged error probability.

In order to compute the minimum averaged error probability, we deform the expression of the optimal POVM by using the tensor product representation of the unitary group  $U(d)$ .

Any irreducible representation of the unitary group  $U(d)$  is characterized by a Young diagram  $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_d]$  and is denoted by  $\mathcal{U}_\lambda$ . We use the shorthand notations  $\lambda_1$  and  $[\lambda_1, \lambda_2]$  to denote  $[\lambda_1, 0, 0, \dots, 0]$  and  $[\lambda_1, \lambda_2, 0, 0, \dots, 0]$ . Note that  $\mathcal{U}_L$  means the totally symmetric subspace of  $(\mathbb{C}^d)^{\otimes L}$ . The dimension of  $\mathcal{U}_{[\lambda_1, \lambda_2]}$  is given as:

$$\dim \mathcal{U}_{[\lambda_1, \lambda_2]} = \frac{(\lambda_1 + d - 1)!(\lambda_2 + d - 2)!(\lambda_1 - \lambda_2 + 1)}{(d - 1)!(d - 2)!(\lambda_1 + 1)!\lambda_2!}. \quad (8)$$

In our problem, the total system size is  $N \equiv M + N_1 + N_2$ , and the total tensor product space  $(\mathbb{C}^d)^{\otimes N}$  can be decomposed to

$$(\mathbb{C}^d)^{\otimes N} = \oplus_{\lambda} \mathcal{U}_\lambda \otimes \mathcal{V}_\lambda, \quad (9)$$

where  $\mathcal{V}_\lambda$  corresponds to the multiplicity of the irreducible space  $\mathcal{U}_\lambda$ .

Since the tensor product space  $(\mathbb{C}^d)^{\otimes N}$  contains the two subspaces  $\mathcal{U}_{N_1} \otimes \mathcal{U}_{M+N_2}$  and  $\mathcal{U}_{M+N_1} \otimes \mathcal{U}_{N_2}$  without multiplicity, these two subspaces have the form:

$$\mathcal{U}_{N_1} \otimes \mathcal{U}_{M+N_2} = \bigoplus_{k=0}^{N_1} \mathcal{U}_{[N-k,k]} \otimes \mathbb{C}|u_k\rangle, \quad (10)$$

$$\mathcal{U}_{M+N_1} \otimes \mathcal{U}_{N_2} = \bigoplus_{k=0}^{\min(M+N_1, N_2)} \mathcal{U}_{[N-k,k]} \otimes \mathbb{C}|v_k\rangle \quad (11)$$

by using two normalized vectors:

$$|u_k\rangle \in \mathcal{V}_{[N-k,k]} \quad (0 \leq k \leq N_1), \quad (12)$$

$$|v_k\rangle \in \mathcal{V}_{[N-k,k]} \quad (0 \leq k \leq \min(M+N_1, N_2)), \quad (13)$$

satisfying  $\langle u_k | v_k \rangle \geq 0$ . Since the dimension of  $\mathcal{V}_N$  is one, the relation  $|u_0\rangle = |v_0\rangle$  holds.

Letting  $I_\lambda$  be the projector onto the space  $\mathcal{U}_\lambda$ , one obtains from the equations (10) and (11)

$$I_{N_1} \otimes I_{M+N_2} = \sum_{k=0}^{N_1} I_{[N-k,k]} \otimes |u_k\rangle\langle u_k|, \quad (14)$$

$$I_{M+N_1} \otimes I_{N_2} = \sum_{k=0}^{\min(M+N_1, N_2)} I_{[N-k,k]} \otimes |v_k\rangle\langle v_k|. \quad (15)$$

Here, we note that the ranges of summations are different from each other. Using these equations, one has

$$\begin{aligned} & \frac{I_{N_1} \otimes I_{M+N_2}}{A_1} - \frac{I_{M+N_1} \otimes I_{N_2}}{A_2} \\ &= \sum_{k=0}^{N_1} I_{[N-k,k]} \otimes \left( \frac{|u_k\rangle\langle u_k|}{A_1} - \frac{|v_k\rangle\langle v_k|}{A_2} \right) \\ & \quad - \sum_{k=N_1+1}^{\min(M+N_1, N_2)} I_{[N-k,k]} \otimes \frac{|v_k\rangle\langle v_k|}{A_2}. \end{aligned} \quad (16)$$

Since  $|u_k\rangle$  and  $|v_k\rangle$  are linearly independent in the range  $1 \leq k \leq N_1$  (we show in Eq. (28)), there exists only one negative eigenvalue of  $\frac{|u_k\rangle\langle u_k|}{A_1} - \frac{|v_k\rangle\langle v_k|}{A_2}$ . Using the normalized eigenvector  $|w_k\rangle \in \mathcal{V}_{[N-k,k]}$  with this eigenvalue, we therefore can write the optimal POVM as

$$\begin{aligned} E_{M, N_1, N_2} &= \sum_{k=1}^{N_1} I_{[N-k,k]} \otimes |w_k\rangle\langle w_k| \\ & \quad + \sum_{k=N_1+1}^{\min(M+N_1, N_2)} I_{[N-k,k]} \otimes |v_k\rangle\langle v_k|. \end{aligned} \quad (17)$$

### III. THE MINIMUM AVERAGED ERROR PROBABILITY

In this section, we will compute the minimum averaged error probability. Plugging equations (16) and (17) into

Eq. (4) as  $E_2 = E_{M, N_1, N_2}$ , one obtains

$$\begin{aligned} & \bar{p}_{M, N_1, N_2}(E_{M, N_1, N_2}) \\ &= \frac{1}{2} \left[ \sum_{k=1}^{N_1} \text{Tr}[I_{[N-k,k]}] \langle w_k | \left( \frac{|u_k\rangle\langle u_k|}{A_1} - \frac{|v_k\rangle\langle v_k|}{A_2} \right) | w_k \rangle \right. \\ & \quad \left. - \frac{1}{A_2} \sum_{k=N_1+1}^{\min(M+N_1, N_2)} \text{Tr}[I_{[N-k,k]}] + 1 \right]. \end{aligned} \quad (18)$$

When arbitrary two real non-zero constants  $C_1$  and  $C_2$  and two linearly independent normalized vectors  $|a\rangle$  and  $|b\rangle$  are given, the unique negative eigenvalue of  $\frac{|a\rangle\langle a|}{C_1} - \frac{|b\rangle\langle b|}{C_2}$  is given by

$$\frac{C_2 - C_1 - \sqrt{(C_2 - C_1)^2 + 4C_1C_2(1 - |\langle a|b\rangle|^2)}}{2C_1C_2}. \quad (19)$$

Therefore, the eigenvector  $|w_k\rangle$  associated with the negative eigenvalue satisfies

$$\begin{aligned} & \langle w_k | \left( \frac{|u_k\rangle\langle u_k|}{A_1} - \frac{|v_k\rangle\langle v_k|}{A_2} \right) | w_k \rangle \\ &= \frac{A_2 - A_1 - \sqrt{(A_2 - A_1)^2 + 4A_1A_2(1 - |\langle u_k|v_k\rangle|^2)}}{2A_1A_2}. \end{aligned} \quad (20)$$

We also obtain the following equations:

$$\text{Tr}[I_{[N-k,k]}] = \dim \mathcal{U}_{[N-k,k]}, \quad (21)$$

$$\sum_{k=0}^{N_1} \text{Tr}[I_{[N-k,k]}] = A_1, \quad (22)$$

$$\sum_{k=0}^{\min(M+N_1, N_2)} \text{Tr}[I_{[N-k,k]}] = A_2. \quad (23)$$

Using these equations, we can write the minimum averaged error probability as

$$\begin{aligned} & \bar{p}_{M, N_1, N_2}(E_{M, N_1, N_2}) = \\ & \frac{1}{4} \left[ \frac{A_1 + A_2}{A_2} \right. \\ & \quad \left. - \sum_{k=0}^{N_1} \frac{\dim \mathcal{U}_{[N-k,k]}}{A_1A_2} \sqrt{(A_2 - A_1)^2 + 4A_1A_2(1 - |\langle u_k|v_k\rangle|^2)} \right]. \end{aligned} \quad (24)$$

Our remained task is to calculate the inner product  $\langle u_k | v_k \rangle$ . When we denote the highest weight vector of the space  $\mathcal{U}_{[N-k,k]}$  by  $|[N-k, k]^d\rangle$ ,  $\langle u_k | v_k \rangle$  is equal to the inner product of  $|[N-k, k]^d\rangle |u_k\rangle$  and  $|[N-k, k]^d\rangle |v_k\rangle$ . We can assume  $d = 2$  without loss of generality since the inner product does not depend on the dimension. Let us fix some notations as follows:

$$\begin{aligned} \mu_0 &\equiv \frac{M}{2}, \mu_1 \equiv \frac{N_1}{2}, \mu_2 \equiv \frac{N_2}{2}, \\ \mu_{01} &\equiv \frac{M+N_1}{2}, \mu_{02} \equiv \frac{M+N_2}{2}, \mu \equiv \frac{N}{2} - k. \end{aligned} \quad (25)$$

Using Wigner's 6j-function [6], we then can write

$$\langle u_k | v_k \rangle = (-1)^{\mu_0 + \mu_1 + \mu_2 + \mu} \sqrt{(2\mu_{01} + 1)(2\mu_{02} + 1)} \begin{Bmatrix} \mu_1 & \mu_0 & \mu_{01} \\ \mu_2 & \mu & \mu_{02} \end{Bmatrix}. \quad (26)$$

Moreover, Wigner's 6j-function can be computed as

$$\begin{aligned} & \begin{Bmatrix} \mu_1 & \mu_0 & \mu_{01} \\ \mu_2 & \mu & \mu_{02} \end{Bmatrix} \\ &= \frac{(-1)^{\mu_0 + \mu_1 + \mu_2 + \mu}}{\sqrt{(2\mu_{01} + 1)(2\mu_{02} + 1)}} \sqrt{\frac{\binom{N_1}{k} \binom{N_2}{k}}{\binom{M+N_1}{k} \binom{M+N_2}{k}}} \end{aligned} \quad (27)$$

(Appendix 1). Thus, one obtains

$$\langle u_k | v_k \rangle = \sqrt{\frac{\binom{N_1}{k} \binom{N_2}{k}}{\binom{M+N_1}{k} \binom{M+N_2}{k}}}, \quad (28)$$

and in order to denote this value we use the notation  $\phi_k$  satisfying

$$\cos \phi_k = \sqrt{\frac{\binom{N_1}{k} \binom{N_2}{k}}{\binom{M+N_1}{k} \binom{M+N_2}{k}}} = \langle u_k | v_k \rangle. \quad (29)$$

Therefore, the minimum averaged error probability can be written as

$$\begin{aligned} & \bar{P}_{M,N_1,N_2}(E_{M,N_1,N_2}) \\ &= \frac{1}{4} \left[ \frac{A_1 + A_2}{A_2} \right. \\ & \quad \left. - \sum_{k=0}^{N_1} \frac{\dim \mathcal{U}_{[N-k,k]}}{A_1 A_2} \sqrt{(A_2 - A_1)^2 + 4A_1 A_2 \sin^2 \phi_k} \right]. \end{aligned} \quad (30)$$

In the case of  $d = 2$ , since

$$\begin{aligned} A_1 &= (N_1 + 1)(M + N_2 + 1) \\ A_2 &= (M + N_1 + 1)(N_2 + 1) \\ A_2 - A_1 &= M(N_2 - N_1) \\ \dim \mathcal{U}_{[N-k,k]} &= M + N_1 + N_2 - 2k + 1, \end{aligned}$$

(30) is calculated to the following way.

$$\begin{aligned} \bar{P}_{M,N_1,N_2}(E_{M,N_1,N_2}) &= \frac{1}{4} \left[ 1 + \frac{(N_1 + 1)(M + N_2 + 1)}{(M + N_1 + 1)(N_2 + 1)} - \sum_{k=0}^{N_1} \frac{M + N_1 + N_2 - 2k + 1}{(N_1 + 1)(M + N_2 + 1)(M + N_1 + 1)(N_2 + 1)} \times \right. \\ & \quad \left. \sqrt{M^2(N_2 - N_1)^2 + 4(N_1 + 1)(M + N_2 + 1)(M + N_1 + 1)(N_2 + 1)} \left[ 1 - \left( \frac{N_1!(M + N_1 - k)!}{(M + N_1)!(N_1 - k)!} \right) \left( \frac{N_2!(M + N_2 - k)!}{(M + N_2)!(N_2 - k)!} \right) \right] \right] \end{aligned} \quad (31)$$

When  $N_1 = N_2$ , the equation  $A_1 = A_2$  holds and this probability is concretely computed as

$$\begin{aligned} & \bar{P}_{M,N_1,N_1}(E_{M,N_1,N_1}) \\ &= \frac{1}{2} \left[ 1 - \frac{(d-1)N_1!(M + N_1)!}{(N_1 + d - 1)!(M + N_1 + d - 1)!} \right. \\ & \quad \times \sum_{k=0}^{N_1} (M + 2N_1 - 2k + 1) \\ & \quad \times \frac{(M + 2N_1 - k + d - 1)!(k + d - 2)!}{(M + 2N_1 - k + 1)!k!} \\ & \quad \left. \times \sqrt{1 - \left( \frac{N_1!(M + N_1 - k)!}{(M + N_1)!(N_1 - k)!} \right)^2} \right]. \end{aligned} \quad (32)$$

Moreover, plugging  $d = 2$  into this equation, we have

$$\begin{aligned} & \bar{P}_{M,N_1,N_1}(E_{M,N_1,N_1}) \\ &= \frac{1}{2} \left[ 1 - \sum_{k=0}^{N_1} \frac{M + 2N_1 - 2k + 1}{(N_1 + 1)(M + N_1 + 1)} \right. \\ & \quad \left. \times \sqrt{1 - \left( \frac{N_1!(M + N_1 - k)!}{(M + N_1)!(N_1 - k)!} \right)^2} \right]. \end{aligned} \quad (33)$$

This result coincides with the result by Sentis et al.[2][9].

#### IV. THE ERROR PROBABILITY WITH THE OPTIMAL POVM

In the previous section, we obtained the averaged error probability when the two candidate states are dis-

tributed independently. However, the unknown states  $\rho_1$  and  $\rho_2$  do not necessarily obey the uniform distribution. In order to treat the performance of our Optimal POVM in a more general setting, we consider the error probability for the given two pure states  $\rho_1$  and  $\rho_2$  when the optimal POVM is applied. This error probability depends on the inner product of two candidates, i.e.,  $q \equiv \text{Tr}[\rho_1 \rho_2]$ . In the following, we calculate the error probability given by Eq. (1) in the case of the optimal POVM  $\{I - E_{M,N_1,N_2}, E_{M,N_1,N_2}\}$ .

**Theorem 1** *The error probability with the optimal POVM  $\{I - E_{M,N_1,N_2}, E_{M,N_1,N_2}\}$  can be written as*

$$p_{M,N_1,N_2}(\rho_1, \rho_2, E_{M,N_1,N_2}) = \frac{1}{4} \sum_{k=0}^{N_1} \left[ P_k + Q_k - \frac{(A_2 - A_1)(P_k - Q_k) + 2 \sin^2 \phi_k (A_1 P_k + A_2 Q_k)}{\sqrt{(A_2 - A_1)^2 + 4 A_1 A_2 \sin^2 \phi_k}} \right], \quad (34)$$

where  $P_k$  and  $Q_k$  is given as follows:

$$P_k \equiv \frac{(N - 2k + 1)N_1!(M + N_2)!}{(N - k + 1)!k!} \times \sum_{l=0}^{N_1-k} \binom{N_1 - k}{l} \binom{M + N_2 - k + l}{l} q^l (1 - q)^{N_1-l}, \quad (35)$$

$$Q_k \equiv \frac{(N - 2k + 1)(M + N_1)!N_2!}{(N - k + 1)!k!} \times \sum_{l=0}^{N_2-k} \binom{M + N_1 - k + l}{l} \binom{N_2 - k}{l} q^l (1 - q)^{N_2-l}. \quad (36)$$

Here we have defined  $0^0 \equiv 1$ .

When  $N_1 = N_2$ , we can write

$$p_{M,N_1,N_1}(\rho_1, \rho_2, E_{M,N_1,N_1}) = \frac{1}{2} \left[ 1 - \sum_{k=0}^{N_1} \frac{(M + 2N_1 - 2k + 1)N_1!(M + N_1)!}{(M + 2N_1 - k + 1)!k!} \times \sqrt{1 - \left( \frac{N_1!(M + N_1 - k)!}{(M + N_1)!(N_1 - k)!} \right)^2} \times \sum_{l=0}^{N_1-k} \binom{N_1 - k}{l} \binom{M + N_1 - k + l}{l} q^l (1 - q)^{N_1-l} \right]. \quad (37)$$

**Proof.** Let us start by defining the notation  $P_k$  and  $Q_k$  as

$$P_k \equiv \text{Tr}[I_{[N-k,k]} \otimes |u_k\rangle\langle u_k| (\rho_1^{\otimes N_1} \otimes \rho_2^{\otimes M+N_2})] \quad (38)$$

$$(0 \leq k \leq N_1),$$

$$Q_k \equiv \text{Tr}[I_{[N-k,k]} \otimes |v_k\rangle\langle v_k| (\rho_1^{\otimes M+N_1} \otimes \rho_2^{\otimes N_2})] \quad (39)$$

$$(0 \leq k \leq \min(M + N_1, N_2)).$$

We note that the following equation holds:

$$\sum_{k=0}^{N_1} P_k = \sum_{k=0}^{\min(M+N_1, N_2)} Q_k = 1. \quad (40)$$

Since an arbitrary pure state  $\sigma$  satisfies

$$I_L^d \sigma^{\otimes L} I_L^d = \sigma^{\otimes L}, \quad (41)$$

one obtains

$$\begin{aligned} & \text{Tr}[(\rho_1^{\otimes N_1} \otimes \rho_2^{\otimes M+N_2}) E_{M,N_1,N_2}] \\ &= \text{Tr}[(I_{N_1} \otimes I_{M+N_2})(\rho_1^{\otimes N_1} \otimes \rho_2^{\otimes M+N_2})(I_{N_1} \otimes I_{M+N_2}) \\ & \quad \times E_{M,N_1,N_2}]. \end{aligned} \quad (42)$$

Plugging equations (14) and (17) into Eq. (42), one has

$$\text{Tr}[(\rho_1^{\otimes N_1} \otimes \rho_2^{\otimes M+N_2}) E_{M,N_1,N_2}] = \sum_{k=1}^{N_1} |\langle v_k | w_k \rangle|^2 P_k. \quad (43)$$

In the same way, using equations (15) and (42), we obtain

$$\begin{aligned} & \text{Tr}[(\rho_1^{\otimes M+N_1} \otimes \rho_2^{\otimes N_2}) E_{M,N_1,N_2}] \\ &= \sum_{k=1}^{N_1} |\langle v_k | w_k \rangle|^2 Q_k + \sum_{k=N_1+1}^{\min(M+N_1, N_2)} Q_k. \end{aligned} \quad (44)$$

When two arbitrary normalized and linearly independent vectors  $|a\rangle, |b\rangle$  and two positive real numbers  $C_1, C_2 > 0$  are given, the normalized eigenvector  $|-\rangle$  with the unique negative eigenvalue of  $\frac{|a\rangle\langle a|}{C_1} - \frac{|b\rangle\langle b|}{C_2}$  satisfies the following equations:

$$|\langle a | - \rangle|^2 = \frac{1}{2} \left[ 1 - \frac{C_2 - C_1 + 2C_1(1 - |\langle a | b \rangle|^2)}{\sqrt{(C_2 - C_1)^2 + 4C_1C_2(1 - |\langle a | b \rangle|^2)}} \right], \quad (45)$$

$$|\langle b | - \rangle|^2 = \frac{1}{2} \left[ 1 - \frac{C_2 - C_1 - 2C_2(1 - |\langle a | b \rangle|^2)}{\sqrt{(C_2 - C_1)^2 + 4C_1C_2(1 - |\langle a | b \rangle|^2)}} \right]. \quad (46)$$

Applying equations (45) and (46) to the case of  $|a\rangle = |u_k\rangle, |b\rangle = |v_k\rangle$ , we have

$$|\langle u_k | w_k \rangle|^2 = \frac{1}{2} \left( 1 - \frac{A_2 - A_1 + 2A_1 \sin^2 \phi_k}{\sqrt{(A_2 - A_1)^2 + 4A_1A_2 \sin^2 \phi_k}} \right), \quad (47)$$

$$|\langle v_k | w_k \rangle|^2 = \frac{1}{2} \left( 1 - \frac{A_2 - A_1 - 2A_2 \sin^2 \phi_k}{\sqrt{(A_2 - A_1)^2 + 4A_1A_2 \sin^2 \phi_k}} \right). \quad (48)$$

for  $1 \leq k \leq N_1$ ,

Using these equations, we can write the error probability as

$$\begin{aligned}
& p_{M,N_1,N_2}(\rho_1, \rho_2, E_{M,N_1,N_2}) \\
&= \frac{1}{2} \left[ 1 + \text{Tr}[(\rho_1^{\otimes N_1} \otimes \rho_2^{\otimes M+N_2}) E_{M,N_1,N_2}] \right. \\
&\quad \left. - \text{Tr}[(\rho_1^{\otimes M+N_1} \otimes \rho_2^{\otimes N_2}) E_{M,N_1,N_2}] \right] \\
&= \frac{1}{2} \left[ 1 + \sum_{k=1}^{N_1} |\langle u_k | w_k \rangle|^2 P_k - \sum_{k=1}^{N_1} |\langle v_k | w_k \rangle|^2 Q_k \right. \\
&\quad \left. - \sum_{k=N_1+1}^{\min(M+N_1, N_2)} Q_k \right] \\
&= \frac{1}{4} \sum_{k=0}^{N_1} \left[ P_k + Q_k \right. \\
&\quad \left. - \frac{(A_2 - A_1)(P_k - Q_k) + 2 \sin^2 \phi_k (A_1 P_k + A_2 Q_k)}{\sqrt{(A_2 - A_1)^2 + 4 A_1 A_2 \sin^2 \phi_k}} \right]. \tag{49}
\end{aligned}$$

Now we turn our attention to computing  $P_k$ . We can assume  $d = 2$  since  $P_k$  does not depend on the dimension. By using Clebsch-Gordan coefficients and the notations given by (25), the projector in Eq. (38) can be written as

$$\begin{aligned}
& I_{[N-k,k]} \otimes |u_k\rangle\langle u_k| \\
&= \sum_{l=0}^{N-2k} \sum_{i=0}^{N_1} \sum_{j=0}^{M+N_2} |\langle \mu : \mu - l | \mu_1 : \mu_1 - i; \mu_{02} : \mu_{02} - j \rangle|^2 \\
&\quad \times |\mu_1 : \mu_1 - i\rangle\langle \mu_1 : \mu_1 - i| \otimes |\mu_{02} : \mu_{02} - j\rangle\langle \mu_{02} : \mu_{02} - j|. \tag{50}
\end{aligned}$$

In the following, we fix the notation  $|\uparrow\rangle$  ( $|\downarrow\rangle$ ) to denote the vector in the space  $\mathbb{C}^2$  whose weight is  $\frac{1}{2}$  ( $-\frac{1}{2}$ ). Without loss of generality, we can assume that  $\rho_2$  is  $|\uparrow\rangle\langle\uparrow|$ . Then, we can write

$$\rho_2^{\otimes M+N_2} = |\mu_{02} : \mu_{02}\rangle\langle \mu_{02} : \mu_{02}| = |\uparrow\rangle\langle\uparrow|^{\otimes M+N_2}. \tag{51}$$

Plugging equations (50) and (51) into Eq. (38), we have

$$\begin{aligned}
P_k &= \sum_{l=0}^{N_1-k} \langle \mu_1 : \mu_1 - k - l | \rho_1^{\otimes N_1} | \mu_1 : \mu_1 - k - l \rangle \\
&\quad \times |\langle \mu : \mu - l | \mu_1 : \mu_1 - k - l; \mu_{02} : \mu_{02} \rangle|^2. \tag{52}
\end{aligned}$$

Moreover, converting the variable  $l$  into  $N_1 - k - l$ , this can be written as

$$\begin{aligned}
P_k &= \sum_{l=0}^{N_1-k} \langle \frac{N_1}{2} : -\frac{N_1}{2} + l | \rho_1^{\otimes N_1} | \frac{N_1}{2} : -\frac{N_1}{2} + l \rangle \\
&\quad \times |\langle \frac{N}{2} - k : \frac{N}{2} - N_1 + l | \frac{N_1}{2} : -\frac{N_1}{2} + l; \frac{M+N_2}{2} : \frac{M+N_2}{2} \rangle|^2. \tag{53}
\end{aligned}$$

We can calculate the Clebsch-Gordan coefficients [6] as

$$\begin{aligned}
& |\langle \frac{N}{2} - k : \frac{N}{2} - N_1 + l | \frac{N_1}{2} : -\frac{N_1}{2} + l; \frac{M+N_2}{2} : \frac{M+N_2}{2} \rangle|^2 \\
&= \frac{(N-2k+1)(N_1-k)!(N_1-l)!}{(N-k+1)!(N_1-k-l)!} \\
&\quad \times \frac{(M+N_2)!(M+N_2-k+l)!}{(M+N_2-k)!k!l!}. \tag{54}
\end{aligned}$$

Denoting  $\rho_1 = |\phi_1\rangle\langle\phi_1|$ , we obtain  $|\langle\phi_1|\uparrow\rangle|^2 = q$ . Thus,

$$\begin{aligned}
& \langle \frac{N_1}{2} : -\frac{N_1}{2} + l | \rho_1^{\otimes N_1} | \frac{N_1}{2} : -\frac{N_1}{2} + l \rangle \\
&= \binom{N_1}{l} |\langle\phi_1|\uparrow\rangle|^{2l} |\langle\phi_1|\downarrow\rangle|^{2(N_1-l)} \\
&= \binom{N_1}{l} q^l (1-q)^{N_1-l}. \tag{55}
\end{aligned}$$

Therefore, we can write  $P_k$  as

$$\begin{aligned}
P_k &\equiv \frac{(N-2k+1)N_1!(M+N_2)!}{(N-k+1)!k!} \\
&\quad \times \sum_{l=0}^{N_1-k} \binom{N_1-k}{l} \binom{M+N_2-k+l}{l} q^l (1-q)^{N_1-l}, \tag{56}
\end{aligned}$$

where we have defined  $0^0$  as 1.

In the same way, one obtains

$$\begin{aligned}
Q_k &\equiv \frac{(N-2k+1)(M+N_1)!N_2!}{(N-k+1)!k!} \\
&\quad \times \sum_{l=0}^{N_2-k} \binom{M+N_1-k+l}{l} \binom{N_2-k}{l} q^l (1-q)^{N_2-l}. \tag{57}
\end{aligned}$$

□

## V. LIMIT OF THE MINIMUM AVERAGED ERROR PROBABILITY

When the both numbers of copies in System 1 and System 2 approach infinitely large, we have perfect knowledge to determine the states in the two systems. In this limit, by using Eq. (32), the probability  $\bar{p}_{M,N_1,N_1}(E_{M,N_1,N_1})$  can be written as

$$\begin{aligned}
& \lim_{N_1 \rightarrow \infty} \bar{p}_{M,N_1,N_1}(E_{M,N_1,N_1}) \\
&= \frac{1}{2} \left[ 1 - 2(d-1) \int_0^1 x \sqrt{1-x^{2M}} (1-x^2)^{d-2} dx \right] \\
&= \frac{1}{2} \left[ 1 - (d-1) \int_0^1 \sqrt{1-x^M} (1-x)^{d-2} dx \right], \tag{58}
\end{aligned}$$

where we have defined  $x = \frac{k}{n}$  and used the Euler-McLaurin summation formula. The case of  $d = 2$  coincides with (18) in Santis et al [2], and the case of  $M = 1$  coincides with (41) in A. Hayashi et al [3].

This result could be easily anticipated from the minimum error probability of the discrimination problem [4]. Recall that the minimum error probability given  $M$  identical copies is  $\frac{1}{2} \left[ 1 - \sqrt{1 - (\text{Tr}[\rho_1 \rho_2])^M} \right]$ . Assuming that  $\rho_1$  and  $\rho_2$  are distributed according to  $\mu_{\Theta_d}$  independently, the average is given by

$$\begin{aligned} & \int_{\Theta_d} \int_{\Theta_d} \frac{1}{2} \left[ 1 - \sqrt{1 - (\text{Tr}[\rho_1 \rho_2])^M} \right] \mu_{\Theta_d}(d\rho_1) \mu_{\Theta_d}(d\rho_2) \\ &= \frac{1}{2} \left[ 1 - 2(d-1) \int_0^{\frac{\pi}{2}} \sqrt{1 - (\cos \theta)^{2M}} (\sin \theta)^{2d-3} \cos \theta d\theta \right] \\ &= \frac{1}{2} \left[ 1 - (d-1) \int_0^1 \sqrt{1 - x^M} (1-x)^{d-2} dx \right]. \end{aligned} \quad (59)$$

Therefore, our optimal measurement can achieve the average performance of two state discrimination under the limit  $N_1 = N_2 \rightarrow \infty$ .

Next, we turn our attention to the complementary case, that is, the number of copies in System 0 is infinitely large. By using Eq. (30), the minimum averaged error probability in this limit can be computed as

$$\lim_{M \rightarrow \infty} \bar{p}_{M, N_1, N_2}(E_{M, N_1, N_2}) = \frac{1}{2^{\binom{N_2+d-1}{d-1}}}. \quad (60)$$

Note that this result is independent of  $N_1$ .

In this limit, we have perfect knowledge of the pure state  $\rho$  in System 0 and this problem is equal to distinguishing two states  $\rho^{\otimes N_1} \otimes \hat{\rho}^{\otimes N_2}$  and  $\hat{\rho}^{\otimes N_1} \otimes \rho^{\otimes N_2}$  in the composite system 12. This problem can be regarded as a generalization of state comparison [8]. As is shown in the following, the minimum error probability for these two states can be obtained with a POVM whose elements are  $\{E_1 = I_1 \otimes (I_2 - \rho^{\otimes N_2}), E_2 = I_1 \otimes \rho^{\otimes N_2}\}$ , where  $I_1$  ( $I_2$ ) is the unit matrix on  $(\mathbb{C}^d)^{\otimes N_1}$  ( $(\mathbb{C}^d)^{\otimes N_2}$ ). Here,  $E_1$  ( $E_2$ ) corresponds to the guess  $\rho^{\otimes N_1} \otimes \hat{\rho}^{\otimes N_2}$  ( $\hat{\rho}^{\otimes N_1} \otimes \rho^{\otimes N_2}$ ). We then can write the error probability as

$$\frac{1}{2} \text{Tr}[\rho^{\otimes N_1} \otimes \hat{\rho}^{\otimes N_2} E_2] + \frac{1}{2} \text{Tr}[\hat{\rho}^{\otimes N_1} \otimes \rho^{\otimes N_2} E_1] = \frac{1}{2} (\text{Tr}[\rho \hat{\rho}])^{N_2}, \quad (61)$$

Thus, the average is computed as follows:

$$\begin{aligned} & \int_{\Theta_d} \int_{\Theta_d} \frac{1}{2} (\text{Tr}[\rho \hat{\rho}])^{N_2} \mu_{\Theta_d}(d\rho) \mu_{\Theta_d}(d\hat{\rho}) \\ &= (d-1) \int_0^{\frac{\pi}{2}} (\cos \theta)^{2N_2} (\sin \theta)^{2d-3} \cos \theta d\theta \\ &= \frac{d-1}{2} \int_0^1 x^{N_2} (1-x)^{d-2} dx \\ &= \frac{1}{2^{\binom{N_2+d-1}{d-1}}}. \end{aligned} \quad (62)$$

Since this value is the same as the expression in Eq. (60), the POVM  $\{E_1 = I_1 \otimes (I_2 - \rho^{\otimes N_2}), E_2 = I_1 \otimes \rho^{\otimes N_2}\}$  realizes the optimal performance.

## VI. EXPONENTIAL DECREASING RATE OF THE ERROR PROBABILITY

When every number of copies  $N_1$ ,  $N_2$  and  $M$  is infinitely large, that is, the states in three systems are perfectly known, the error probability approaches zero unless  $q = 1$ . In order to treat the convergence speed, we focus on the exponential decreasing rate of the error probability when the optimal POVM  $E_{M, N_1, N_1}$  is applied. For simplicity, we assume that the numbers  $N_1$ ,  $N_2$  of copies in Systems 1, 2 increase in proportion to the number  $M$  of copies in System 0. When the proportional constant is given to be  $\alpha > 0$ , we have  $N_1 = N_2 = \alpha M$ . Thus, using the real numbers

$$\begin{aligned} C_k &\equiv \frac{(M + 2\alpha M - 2k + 1)(\alpha M)!(M + \alpha M)!}{2(M + 2\alpha M - k + 1)!k!} \\ &\times \left[ 1 - \sqrt{1 - \left( \frac{(\alpha M)!(M + \alpha M - k)!}{(M + \alpha M)!(\alpha M - k)!} \right)^2} \right], \end{aligned} \quad (63)$$

$$D_{k,l} \equiv \binom{\alpha M - k}{l} \binom{M + \alpha M - k + l}{l} q^l (1-q)^{\alpha M - l}, \quad (64)$$

we can write from Eq. (37)

$$p_{M, \alpha M, \alpha M}(\rho_1, \rho_2, E_{M, \alpha M, \alpha M}) = \sum_{k=0}^{\alpha M} \sum_{l=0}^{\alpha M - k} C_k D_{k,l}. \quad (65)$$

The convergence speed is represented as  $\lim_{M \rightarrow \infty} \frac{-1}{M} \log p_{M, \alpha M, \alpha M}(\rho_1, \rho_2, E_{M, \alpha M, \alpha M})$ , which can be deformed as

$$\begin{aligned} & \lim_{M \rightarrow \infty} \frac{-1}{M} \log p_{M, \alpha M, \alpha M}(\rho_1, \rho_2, E_{M, \alpha M, \alpha M}) \\ &= \lim_{M \rightarrow \infty} \frac{-1}{M} \log \left( \sum_{k=0}^{\alpha M} \sum_{l=0}^{\alpha M - k} C_k D_{k,l} \right) \\ &= \lim_{M \rightarrow \infty} \frac{-1}{M} \log \left( \max_{0 \leq k \leq \alpha M} \max_{0 \leq l \leq \alpha M - k} C_k D_{k,l} \right). \end{aligned} \quad (66)$$

Moreover, using the approximation formula  $\sqrt{1-x} \approx 1 - \frac{1}{2}x$  (when  $x \ll 1$ ) and the Stirling approximation  $n! \approx n^n e^{-n} \sqrt{2\pi n}$ , one obtains

$$\begin{aligned} & \lim_{M \rightarrow \infty} \frac{-1}{M} \log p_{M, \alpha M, \alpha M}(\rho_1, \rho_2, E_{M, \alpha M, \alpha M}) \\ &= \min_{0 \leq \beta \leq \alpha} \min_{0 \leq \gamma \leq \alpha - \beta} h(\beta, \gamma), \end{aligned} \quad (67)$$

where we have defined as  $\beta \equiv \frac{k}{M}, \gamma \equiv \frac{l}{M}$  and

$$\begin{aligned}
h(\beta, \gamma) \equiv & (\alpha - \beta - \gamma) \log(\alpha - \beta - \gamma) \\
& - (1 + \alpha - \beta + \gamma) \log(1 + \alpha - \beta + \gamma) + (\alpha - \beta) \log(\alpha - \beta) \\
& - (1 + \alpha - \beta) \log(1 + \alpha - \beta) - 3\alpha \log \alpha \\
& + (1 + \alpha) \log(1 + \alpha) + \beta \log \beta \\
& + (1 + 2\alpha - \beta) \log(1 + 2\alpha - \beta) \\
& + 2\gamma \log \gamma - \gamma \log q - (\alpha - \gamma) \log(1 - q).
\end{aligned} \tag{68}$$

There is the unique root of  $\frac{\partial h}{\partial \beta} = \frac{\partial h}{\partial \gamma} = 0$  in the range  $0 < \beta < \alpha, 0 < \gamma < \alpha - \beta$  and we use  $(\beta_1, \gamma_1)$  to denote it. These can be calculated as

$$\gamma_1 = \frac{q(\alpha - 1) + \sqrt{q^2(\alpha - 1)^2 + 4q\alpha}}{2}, \tag{69}$$

$$\beta_1 = \frac{(2\alpha + 1) - \sqrt{(2\alpha + 1)^2 - 4(\alpha^2 - \alpha\gamma_1)}}{2}. \tag{70}$$

One can agree that  $h(\beta_1, \gamma_1)$  is the minimum of the function  $h$  in the range  $0 < \beta < \alpha, 0 < \gamma < \alpha - \beta$  due to the following equations:

$$\begin{aligned}
\frac{\partial^2 h}{\partial \beta^2}(\beta, \gamma) &> 0, \\
\frac{\partial^2 h}{\partial \gamma^2}(\beta, \gamma) &> 0, \\
\lim_{\beta \rightarrow 0} \frac{\partial h}{\partial \beta}(\beta, \gamma) &= \lim_{\gamma \rightarrow 0} \frac{\partial h}{\partial \gamma}(\beta, \gamma) = -\infty, \\
\lim_{\beta \rightarrow \alpha - \gamma} \frac{\partial h}{\partial \beta}(\beta, \gamma) &= \lim_{\gamma \rightarrow \alpha - \beta} \frac{\partial h}{\partial \gamma}(\beta, \gamma) = +\infty,
\end{aligned} \tag{71}$$

When  $\alpha$  is sufficient large, we can write

$$h(\beta_1, \gamma_1) = -\log q - \frac{1 - q}{q(\alpha - 1) + 2} + O\left(\frac{1}{\alpha^2}\right). \tag{72}$$

In fact, as is numerically demonstrated in Figs. 2 and 3,  $-\log q - \frac{1 - q}{q(\alpha - 1) + 2}$  well approximates  $h(\beta_1, \gamma_1)$  when  $\alpha$  is large. Therefore, we obtain the convergence speed of the error probability,

$$\begin{aligned}
& \lim_{M \rightarrow \infty} \frac{-1}{M} \log p_{M, \alpha M, \alpha M}(\rho_1, \rho_2, E_{M, \alpha M, \alpha M}) \\
&= -\log \text{Tr}[\rho_1 \rho_2] - \frac{1 - \text{Tr}[\rho_1, \rho_2]}{\text{Tr}[\rho_1 \rho_2](\alpha - 1) + 2} + O\left(\frac{1}{\alpha^2}\right).
\end{aligned} \tag{73}$$

In the discrimination problem of two pure states [4], when the number of copies of the state to be identified is infinitely large, the convergence speed is given by

$$\lim_{M \rightarrow \infty} \frac{-1}{M} \log \frac{1}{2} \left[ 1 - \sqrt{1 - (\text{Tr}[\rho_1 \rho_2])^M} \right] = -\log \text{Tr}[\rho_1 \rho_2]. \tag{74}$$

This is called the quantum Chernoff bound [5] and equal to the limit of Eq. (73) as  $\alpha \rightarrow \infty$ . This fact means that the performance of our optimal POVM is close to that of the optimal POVM in the sense of quantum state discrimination.

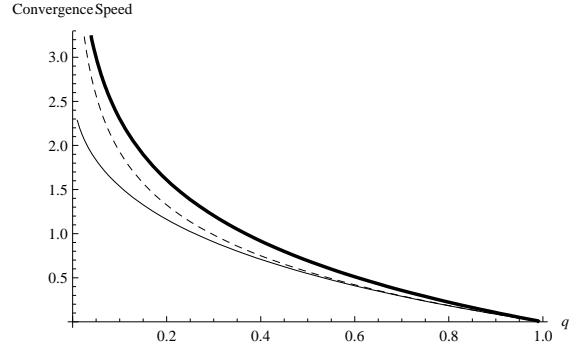


FIG. 2: (normal line)  $h(\beta_1, \gamma_1)$ , (dashed line)  $-\log q - \frac{1 - q}{q(\alpha - 1) + 2}$ , (thick line)  $-\log q$  for  $\alpha = 5$

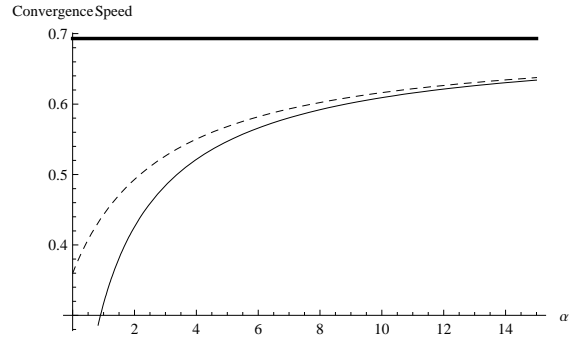


FIG. 3: (normal line)  $h(\beta_1, \gamma_1)$ , (dashed line)  $-\log q - \frac{1 - q}{q(\alpha - 1) + 2}$ , (thick line)  $-\log q$  for  $q = 0.5$

## VII. CONCLUSIONS

We have studied the changepoint problem in the quantum setting, where our task is to choose the correct changepoint between two candidates. This problem is equal to discriminating two unknown general states when multiple copies of the state are provided. We have obtained the minimum averaged error probability, Eq. (30). Our result of special cases coincides with the results by [2] and [3]. However, when arbitrary numbers of copies of general pure states are given, we have calculated it for the first time. Moreover, we have first calculated the non-averaged error probability Eq.(34). This depends on the inner product. As could be anticipated, when the numbers of copies of candidates are infinitely large, we recover the average of the usual discrimination problem. We have also paid attention to the exponential decreasing rate and shown the convergence rate of the non-averaged error probability approaches the quantum Chernoff bound.



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### Appendix A: Calculation of Wigner’s 6j-function

Let us consider the calculation of the Wigners 6j-function  $\left\{ \begin{smallmatrix} a & b & c \\ d & e & f \end{smallmatrix} \right\}$ . Let us define some notations as

$$\begin{aligned} \alpha_1 &\equiv a + b + d + e, \alpha_2 \equiv a + c + d + f, \alpha_3 \equiv b + c + e + f, \\ \beta_1 &\equiv a + b + c, \beta_2 \equiv a + e + f, \\ \beta_3 &\equiv b + d + f, \beta_4 \equiv c + d + e, \end{aligned} \quad (\text{A1})$$

and let us define  $A_1, A_2$  and  $A_3$  to be the smallest, middle, and largest values of  $\alpha_1, \alpha_2$  and  $\alpha_3$  and  $B_1, B_2, B_3$  and  $B_4$  to be the smallest, second smallest, second largest, and largest values of  $\beta_1, \beta_2, \beta_3$  and  $\beta_4$ . When  $B_4 = A_1$ , from the formula in [7] we can calculate as

$$\begin{aligned} \left\{ \begin{smallmatrix} a & b & c \\ d & e & f \end{smallmatrix} \right\} &= (-1)^{B_4} \left[ \prod_{i=1}^{B_4-B_3} \frac{(B_3+1+i)(A_3-A_1+i)}{(A_3-B_1+i)(A_2-B_2+i)} \right]^{\frac{1}{2}} \\ &\times \left[ \frac{1}{(B_1+1)(B_2+1)} \right] \\ &\times \prod_{i=1}^{A_2-A_1} \frac{(A_1-B_1+i)(B_4-B_2+i)(B_4-B_3+i)}{(A_1+B_1-A_2+i)(A_1+B_2-A_2+i)i} \right]^{\frac{1}{2}}. \end{aligned} \quad (\text{A2})$$

We now compute the Wignerfs 6-j function  $\left\{ \begin{smallmatrix} \mu_1 & \mu_0 & \mu_{01} \\ \mu_2 & \mu & \mu_{02} \end{smallmatrix} \right\}$  in Eq.(26).

**In case  $0 \leq k \leq M$**

In this case, we can write as

$$\begin{aligned} A_1 &= N - k, A_2 = N, A_3 = M + N - k \\ B_1 &= M + N_1, B_2 = M + N_2, B_3 = N - k, B_4 = N - k. \end{aligned} \quad (\text{A3})$$

Plugging these into Eq. (A2), one obtains

$$\begin{aligned} \left\{ \begin{smallmatrix} \mu_1 & \mu_0 & \mu_{01} \\ \mu_2 & \mu & \mu_{02} \end{smallmatrix} \right\} &= (-1)^{N-k} \left[ \frac{1}{(M+N_1+1)(M+N_2+1)} \right] \\ &\times \prod_{i=1}^k \frac{(N_1-k+i)(N_2-k+i)}{(M+N_1-k+i)(M+N_2-k+i)} \right]^{\frac{1}{2}}. \end{aligned} \quad (\text{A4})$$

This is deformed as

$$\begin{aligned} &\left\{ \begin{smallmatrix} \mu_1 & \mu_0 & \mu_{01} \\ \mu_2 & \mu & \mu_{02} \end{smallmatrix} \right\} \\ &= \frac{(-1)^{\mu_0+\mu_1+\mu_2+\mu}}{\sqrt{(2\mu_{01}+1)(2\mu_{02}+1)}} \sqrt{\frac{\binom{N_1}{k} \binom{N_2}{k}}{\binom{M+N_1}{k} \binom{M+N_2}{k}}}. \end{aligned} \quad (\text{A5})$$

**In case  $M+1 \leq k \leq N_1$**

In this case, we can write as

$$\begin{aligned} A_1 &= N - k, A_2 = M + N - k, A_3 = N \\ B_1 &= M + N_1, B_2 = M + N_2, B_3 = N - k, B_4 = N - k. \end{aligned} \quad (\text{A6})$$

Plugging these into Eq. (A2), one obtains

$$\begin{aligned} \left\{ \begin{smallmatrix} \mu_1 & \mu_0 & \mu_{01} \\ \mu_2 & \mu & \mu_{02} \end{smallmatrix} \right\} &= (-1)^{N-k} \left[ \frac{1}{(M+N_1+1)(M+N_2+1)} \right] \\ &\times \prod_{i=1}^M \frac{(N_1-k+i)(N_2-k+i)}{(N_1+i)(N_2+i)} \right]^{\frac{1}{2}}. \end{aligned} \quad (\text{A7})$$

Since  $M+1 \leq k$ , one has

$$\frac{\binom{N_1}{k} \binom{N_2}{k}}{\binom{M+N_1}{k} \binom{M+N_2}{k}} = \prod_{i=1}^M \frac{(N_1-k+i)(N_2-k+i)}{(N_1+i)(N_2+i)}. \quad (\text{A8})$$

Thus,

$$\begin{aligned} &\left\{ \begin{smallmatrix} \mu_1 & \mu_0 & \mu_{01} \\ \mu_2 & \mu & \mu_{02} \end{smallmatrix} \right\} \\ &= \frac{(-1)^{\mu_0+\mu_1+\mu_2+\mu}}{\sqrt{(2\mu_{01}+1)(2\mu_{02}+1)}} \sqrt{\frac{\binom{N_1}{k} \binom{N_2}{k}}{\binom{M+N_1}{k} \binom{M+N_2}{k}}}. \end{aligned} \quad (\text{A9})$$

also holds.

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  - [9] Sentis et al.[2] also treated the case of  $N_1 \neq N_2$ . In their derivation, they simply apply (10) of [2] with the inner product (28) to each irreducible component. It can be applied only when  $A_1 = A_2$ . However, in the case of  $N_1 \neq N_2$ , the equation  $A_1 = A_2$  does not hold. Hence, their result (A2) is valid only when  $N_1 = N_2$ . In our derivation, we apply the formula (20) instead of their (10). In fact, their result (A2) is different from our result (31).